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In Metric-measure Spaces Sobolev Embedding is Equivalent to a Lower Bound for the Measure

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Abstract We study Sobolev inequalities on doubling metric measure spaces. We investigate the relation between Sobolev embeddings and lower bound for measure. In particular, we prove that if the Sobolev inequality holds, then the measure μ satisfies the lower bound, i.e. there exists b such that $\mu(B(x,r)) \geq br^{\alpha}$ for $r \in (0,1]$ and any point x from metric space.

Keywords Sobolev spaces · Sobolev inequalities · Metric measure spaces · Lower bound for measure

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1 Introduction

Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$. If the boundary of Ω is sufficiently regular and $1 \leq p < n$, then the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ holds, where $p^* := \frac{np}{n-p}$ (see e.g. [1]). On the other hand, it was shown by Hajłasz-Koskela-Tuominen [7] that if $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, then Ω satisifies the so-called measure density condition, i.e. there exists a constant $c >$ such that for all $x \in \Omega$ and all $0 < r \leq 1$

$$|B(x,r) \cap \Omega| \geq cr^n. \tag{1}$$

More recently, the result of Hajłasz-Koskela-Tuominen has been extended to the Slobodeckij-Sobolev spaces $W^{s,p}$ (see Zhou [12]). Namely, if $W^{s,p}(\Omega) \hookrightarrow L^{np/(n-sp)}(\Omega)$ for some $s \in (0,1)$ and $p \geq 1$ such that $sp < n$, then Ω satisfies (1).

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The main objective of the paper is to study the relation between Sobolev inequalities on metric measure spaces and lower bound for measure. In particular, we prove the following result. Suppose that (X, ρ, μ) is a metric measure space equipped with doubling measure and let $M^{1,p}(X)$ be the Hajłasz-Sobolev space. If $M^{1,p}(X) \hookrightarrow L^q(X)$, where $p < q$, then there exists $b > 0$ such that for any $x \in X$ and $0 < r \leq 1$, the following inequality holds

$$\mu(B(x, r)) \geq br^{\alpha},$$

where $\frac{1}{p} - \frac{1}{q} = \frac{1}{\alpha}$.

The remainder of the paper is structured as follows. In Section 2, we introduce the notations and recall the notion of Sobolev spaces on general metric measure spaces. Our principal assertion, concerning the sufficient and necessary conditions for Sobolev embeddings are formulated and proven in Section 3.

2 Preliminaries

Let (X, ρ, μ) be a metric measure space equipped with a metric ρ and the Borel regular measure μ. We assume throughout the paper that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. Additionally, we assume that the measure μ satisfies a doubling condition. It means that, there exists a constant $C_d > 0$ such that for every ball $B(x, r)$,

$$\mu\left(B(x, 2r)\right) \leq C_d \mu\left(B(x, r)\right).$$

It is well known (see e.g. Lemma 8.1.13 in [9]) that the doubling condition implies that, there exists a positive constant D satisfying

$$D\left(\frac{r_1}{r_2}\right)^s \leq \frac{\mu\left(B(x_1, r_1)\right)}{\mu\left(B(x_2, r_2)\right)}, \quad \text{where} \quad s = \log_2 C_d,$$

for all balls $B(x_2, r_2)$ and $B(x_1, r_1)$ with $r_2 \geq r_1 > 0$ and $x_1 \in B(x_2, r_2)$. It follows from the above inequality that if X is bounded, then there exists $b > 0$ such that the following inequality holds for $r < \operatorname{diam} X$

$$\mu(B(x, r)) \geq br^s. \tag{2}$$

On the other hand, if the metric measure space equipped with a doubling measure is not bounded, then inequality (2) does not necessarily hold.

Furthermore, we need to recall the notion of Ahlfors regularity. We shall say that the metric measure space (X, ρ, μ) is Ahlfors s-regular if there exist constants b and B such that

$$br^s \leq \mu\left(B(x, r)\right) \leq Br^s$$

for all balls $B(x, r) \subset X$ with $r < \operatorname{diam} X$.

We are now in a position to recall the notion of Sobolev spaces on metric measure spaces (see also [4]). Let (X, ρ, μ) be a metric measure space. We say that a p-integrable function f belongs to the Hajłasz-Sobolev space $M^{1,p}(X)$ if there exists non-negative $g \in L^p(X)$, called a generalized gradient, such that

$$|f(x) - f(y)| \leq \rho(x, y)\left(g(x) + g(y)\right) \quad \text{a.e. for} \quad x, y \in X.$$

We equip the space $M^{1,p}(X)$ with the norm

$$\|f\|_{M^{1,p}(X)} = \|f\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all the generalized gradients. Then $M^{1,p}$ is a Banach space. For the basic properties of this kind of spaces, we refer to [2, 4–6, 9–11].

Suppose that f is locally integrable and A is a measurable set of positive measure, then by f_A we denote the integral average of the function f over the set A, i.e.,

$$f_A := \fint_A f d\mu = \frac{1}{\mu(A)} \int_A f d\mu.$$

3 Main Results

In this section, we show the sufficient and necessary conditions for Sobolev inequalities. We will start with the following proposition.

Proposition 3.1 *Let (X, ρ, μ) be a metric measure space with s-regular measure μ. If $s > p \geq 1$, then*

$$M^{1,p}(X) \hookrightarrow L^{p^*}(X),$$

where $p^* = \frac{sp}{s-p}$. Moreover, there exists $C = C(s, p, b)$, depending on s, p, b, such that for each $u \in M^{1,p}(X)$, the following inequality holds

$$\|u\|_{L^{p^*}(X)} \leq C \left(\|u\|_{L^p(X)} + \|g\|_{L^p(X)}\right).$$

Furthermore, if $\operatorname{diam} X = \infty$*, then*

$$\|u\|_{L^{p^*}(X)} \leq C\|g\|_{L^p(X)}.$$

Proof Taking $\sigma = 2$ in Theorem 8.7 from [5], we have

$$\left(\fint_{B(x_0,r)} \left|u - u_{B(x_0,r)}\right|^{p^*} d\mu\right)^{\frac{1}{p^*}} \leq C \left(\frac{\mu(B(x_0, 2r))}{br^s}\right)^{\frac{1}{p}} r \left(\fint_{B(x_0,2r)} g^p d\mu\right)^{\frac{1}{p}},$$

where the constant C depends on p and s. Hence, we get

$$\begin{aligned}\left(\int_{B(x_0,r)} |u|^{p^*} d\mu\right)^{\frac{1}{p^*}} &\leq \left(\int_{B(x_0,r)} |u_{B(x_0,r)}|^{p^*} d\mu\right)^{\frac{1}{p^*}} \\ &\quad + Cb^{-\frac{1}{p}} \frac{\mu(B(x_0, r))^{\frac{1}{p^*}}}{r^{\frac{s}{p^*}}} \left(\int_{B(x_0,2r)} g^p d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}\left(\int_{B(x_0,r)} |u|^{p^*} d\mu\right)^{\frac{1}{p^*}} &\leq \frac{b^{-\frac{1}{s}}}{r} \left(\int_{B(x_0,r)} |u|^p d\mu\right)^{\frac{1}{p}} + Cb^{-\frac{1}{p}} B^{\frac{1}{p^*}} \left(\int_{B(x_0,2r)} g^p d\mu\right)^{\frac{1}{p}} \leq \\ &\quad \frac{b^{-\frac{1}{s}}}{r} \left(\int_X |u|^p d\mu\right)^{\frac{1}{p}} + Cb^{-\frac{1}{p}} B^{\frac{1}{p^*}} \left(\int_X g^p d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

Finally, by passing to $\operatorname{diam} X$ with r, we obtain the desired result. □

Next, we state necessary conditions for Sobolev embeddings. The proof of the following theorem relies on the methods established by Carron (see [3] and proof of Lemma 2.2 in [8]).

Theorem 3.2 *Suppose that (X, ρ, μ) is a metric measure space with the doubling measure. If*

$$M^{1,p}(X) \hookrightarrow L^q(X),$$

where $q > p$, then there exists $b = b(p, q, C_{pq})$ such that

$$\mu(B(x, r)) \geq br^{\alpha}, \quad \text{for} \quad r \in (0, 1],$$

where $\frac{1}{p} - \frac{1}{q} = \frac{1}{\alpha}$ and $C_{p,q}$ is the constant of the embedding.

Remark 1 Let us stress that b does not depend on the doubling constant C_d.

Proof For each $u \in M^{1,p}(X)$ we have

$$\left(\int_X |u|^q \, d\mu\right)^{\frac{1}{q}} \leq C_{pq} \left(\left(\int_X |u|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p \, d\mu\right)^{\frac{1}{p}}\right),$$

where g is a generalized gradient of u. For a fixed $x \in X$ and $R > 0$, let us define a Lipschitz function u_R as follows

$$u_R(y) := \begin{cases} \frac{2}{R}(R - \rho(y, x)) & \text{if } y \in B(x, R) \setminus B\left(x, \frac{R}{2}\right) \\ 1 & \text{if } y \in B(x, \frac{R}{2}) \\ 0 & \text{if } y \in X \setminus B(x, R) \,, \end{cases}$$

It is easily seen that as a generalized gradient we can take

$$g_R(y) = \frac{2}{R} \chi_{B(x,R)}.$$

Since the support of u is contained in $B(x, R)$, we get in view of the Hölder inequality

$$\left(\int_X |u_R|^p \, d\mu\right)^{\frac{1}{p}} \leq \mu(B(x, R))^{\frac{1}{\alpha}} \left(\int_X |u_R|^q \, d\mu\right)^{\frac{1}{q}}.$$

Therefore, we have

$$\frac{1}{\mu(B(x, R))^{\frac{1}{\alpha}}} - C_{pq} \leq C_{pq} \frac{\left(\int_X g_R^p \, d\mu\right)^{\frac{1}{p}}}{\left(\int_X |u_R|^p \, d\mu\right)^{\frac{1}{p}}}. \tag{3}$$

Let us fix $r \leq 1$ and $x \in X$. Then, $\mu(B(x, r)) \geq \left(\frac{1}{2C_{pq}}\right)^{\alpha} r^{\alpha}$ or $\mu(B(x, r)) \leq \left(\frac{1}{2C_{pq}}\right)^{\alpha} r^{\alpha}$. In the first case we have the desired inequality. Thus we may assume that $\mu(B(x, r)) \leq \left(\frac{1}{2C_{pq}}\right)^{\alpha} r^{\alpha}$. In this case, we get for any $\delta \leq r$

$$\mu(B(x, \delta)) \leq \left(\frac{1}{2C_{pq}}\right)^{\alpha}.$$

Due to (3), we have

$$\frac{1}{(2C_{pq})^p} \mu(B(x, \delta))^{-\frac{p}{\alpha}} \leq \frac{\int_X g_\delta^p \, d\mu}{\int_X |u_\delta|^p \, d\mu}.$$

Consequently, the structure of u_δ and g_δ implies the following inequality

$$\frac{1}{(2C_{pq})^p} \mu(B(x, \delta))^{-\frac{p}{\alpha}} \leq \frac{\left(\frac{2}{\delta}\right)^p \mu(B(x, \delta))}{\mu(B(x, \frac{\delta}{2}))}.$$

Hence, we obtain the following estimate

$$\mu(B(x,\delta)) \geq \left(\frac{\delta}{4C_{pq}}\right)^{\frac{\alpha p}{p+\alpha}} \left(\mu\left(B\left(x,\frac{\delta}{2}\right)\right)\right)^{\frac{\alpha}{p+\alpha}} \tag{4}$$

for each $\delta \leq r$. Thus, Theorem 3.2 easily follows from the above estimate and the doubling condition, but we wish to obtain the constant b independent of the doubling constant C_d, so more work is required.

In order to reach our goal, we iterate inequality (4) and we get for any integer n

$$\mu(B(x,r)) \geq \left(\frac{r}{2C_{pq}}\right)^{p\sum_{j=1}^{n}\left(\frac{\alpha}{p+\alpha}\right)^j} \left(\frac{1}{2}\right)^{p\sum_{j=1}^{n} j\left(\frac{\alpha}{p+\alpha}\right)^j} \left(\mu\left(B\left(x,\frac{r}{2^n}\right)\right)\right)^{\left(\frac{\alpha}{p+\alpha}\right)^n}. \tag{5}$$

On the other hand, since the measure is doubling, we have

$$(\mu\left(B\left(x,r\right)\right))^{\left(\frac{\alpha}{p+\alpha}\right)^n} \geq \left(\mu\left(B\left(x,\frac{r}{2^n}\right)\right)\right)^{\left(\frac{\alpha}{p+\alpha}\right)^n} \geq C_d^{-n\left(\frac{\alpha}{p+\alpha}\right)^n} (\mu\left(B\left(x,r\right)\right))^{\left(\frac{\alpha}{p+\alpha}\right)^n},$$

hence

$$\lim_{n\to\infty}\left(\mu\left(B\left(x,\frac{r}{2^n}\right)\right)\right)^{\left(\frac{\alpha}{p+\alpha}\right)^n} = 1.$$

Thus, letting $n \to \infty$ in (5), we obtain

$$\mu(B(x,r)) \geq \frac{1}{\left(2C_{pq}\right)^{\alpha} 2^{\frac{\alpha(\alpha+p)}{p}}} r^{\alpha}.$$

This completes the proof of our claim. □

Now, as a corollary from the proof of Theorem 3.2 and Proposition 3.1, we obtain the following theorem.

Theorem 3.3 *Suppose that (X,ρ,μ) is an unbounded metric measure space with the doubling measure such that $\mu(B(x,r)) \leq Br^{\alpha}$ for all $r > 0$ and $x \in X$. Assume that $1 \leq p < q$ and $\alpha = \frac{pq}{q-p}$. Then, the following conditions are equivalent:*

i) *there exists $b > 0$ such that for any $x \in X$ and $r > 0$, the following inequality holds*

$$\mu(B(x,r)) \geq br^{\alpha},$$

ii)

$$M^{1,p}(X) \hookrightarrow L^q(X),$$

and, there exists $C > 0$ such that for each $u \in M^{1,p}(X)$ the following inequality holds

$$\|u\|_{L^q(X)} \leq C\|g\|_{L^p(X)}.$$

Proof Implication $i)$ to $ii)$ follows directly from Proposition 3.1. Now, we shall prove the converse. Let us assume that $ii)$ holds. In other words, for any $u \in M^{1,p}(X)$ we have

$$\left(\int_X |u|^q\,d\mu\right)^{\frac{1}{q}} \leq C\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}},$$

where g is a generalized gradient of u. Next, we use the techniques from the proof of Theorem 3.2. We have

$$\frac{1}{\mu(B(x,r))^{\frac{1}{\alpha}}} \leq C \frac{\left(\int_X g_r^p \, d\mu\right)^{\frac{1}{p}}}{\left(\int_X |u_r|^p \, d\mu\right)^{\frac{1}{p}}}.$$

Hence, we obtain the following estimate

$$\mu(B(x,r)) \geq \left(\frac{r}{2C}\right)^{\frac{\alpha p}{p+\alpha}} \left(\mu\left(B\left(x, \frac{r}{2}\right)\right)\right)^{\frac{\alpha}{p+\alpha}}.$$

Iteration of the above inequality leads us to the estimate

$$\mu(B(x,r)) \geq \left(\frac{r}{C}\right)^{p\sum_{j=1}^{n}\left(\frac{\alpha}{p+\alpha}\right)^j} \left(\frac{1}{2}\right)^{p\sum_{j=1}^{n} j\left(\frac{\alpha}{p+\alpha}\right)^j} \left(\mu\left(B\left(x, \frac{r}{2^n}\right)\right)\right)^{\left(\frac{\alpha}{p+\alpha}\right)^n}.$$

Finally, letting $n \to \infty$ in the previous inequality, we get

$$\mu(B(x,r)) \geq \frac{1}{(C)^{\alpha}\, 2^{\frac{\alpha(\alpha+p)}{p}}} r^{\alpha},$$

which completes the proof of Theorem 3.3. □

Furthermore, we have the following equivalence.

Theorem 3.4 *Suppose that (X, ρ, μ) is a bounded metric measure space with the doubling measure. Assume that $1 \leq p < q$ and $\alpha = \frac{pq}{q-p}$. Then, the following conditions are equivalent:*

i) *there exists $b > 0$ such that for any $x \in X$ and $0 < r < \operatorname{diam} X$, the following inequality holds*

$$\mu(B(x,r)) \geq b r^{\alpha},$$

ii)

$$M^{1,p}(X) \hookrightarrow L^q(X).$$

Proof Implication from $i)$ to $ii)$ follows from Theorem 6 in [4]. Next, let us assume that $M^{1,p}(X) \hookrightarrow L^q(X)$, then from Theorem 3.2 we have

$$\mu(B(x,r)) \geq b r^{\alpha}, \quad \text{for} \quad r \in (0,1].$$

From the above estimate, we get for $0 < r < \operatorname{diam} X$ the following inequality

$$\mu(B(x,r)) \geq \mu\left(B\left(x, \frac{r}{\operatorname{diam} X + 1}\right)\right) \geq b\left(\frac{r}{\operatorname{diam} X + 1}\right)^{\alpha} = C r^{\alpha},$$

and the proof is complete. □

Open problem 1 The doubling condition plays a role in the proofs of Theorem 3.2 and Theorem 3.3, but not in the statements of the results. Hence, it is natural to state the following problem. Are the results true without the doubling condition?

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